Measure Theory with Ergodic Horizons
Lecture 26

Hahn Decomposition Theorem. For any signed measure ξ on a measurable space $(X, \mathcal{B})$ there is a partition $X = X_+ \sqcup X_-$, $X_\pm \in \mathcal{B}$, such that $\xi|_{X_+}$ and $-\xi|_{X_-}$ are measures.

Proof of Hahn decomposition. We need to partition X into purely positive and purely negative parts. Assume WLOG that $\xi < \infty$.

Claim. Every non-null positive set $P \subseteq X$ contains a non-null purely positive set $P_+ \subseteq P$ with $\xi(P_+) \geq \xi(P) > 0$.

It remains to collect all purely positive sets into one via $\frac{1}{2}$ measure exhaustion. Given pairwise disjoint sequence $(P_i)_{i<n}$ of purely positive sets, we obtain let P_n be $\geq \frac{1}{2}$ largest purely positive set disjoint from $\bigcup_{i<n} P_i$, i.e.

$$\xi(P_n) \geq \frac{1}{2} \sup\{\xi(P) : P \subseteq X \setminus \bigcup_{i<n} P_i \text{ is purely positive}\}.$$

Thus, we obtain a sequence $(P_n)_{n \in \mathbb{N}}$ of purely positive disjoint sets, so $X_+ := \bigcup_{n \in \mathbb{N}} P_n$ is purely positive. Recall that $\xi(X_+) < \infty$ by our assumption, hence $(\mu(P_n))$ is summable, in particular $\mu(P_n) \to 0$. This implies that X_- is purely negative: if there were a non-null positive set $P \subseteq X_-$, then the claim would give a purely positive non-null set $P^+ \subseteq P$ and taking a large enough n, we'd have $\xi(P_n) < \frac{1}{2}\xi(P^+)$, contradicting the choice of P_n. $\square$

Remark. By definition, a signed measure ξ doesn't attain the value $+\infty$ or $-\infty$, however the fact that $\xi < \infty$ doesn't immediately imply that ξ is bounded above. Nevertheless, the Hahn decomposition implies this. Indeed:

$$\xi(X_-) \le \xi \le \xi(X_+).$$

We are now ready to prove the key lemma towards the Radon-Nikodym theorem.

actually, enough to assume ν is finite and μ is σ-finite

Lemma. Let μ and ν be finite measures on a measurable space $(X, \mathcal{B})$. Then:

either: $\mu \perp \nu$

or: $\mu|_A \ge \varepsilon \cdot \nu|_A$ for some $\varepsilon > 0$ and ν-positive measure set $A \in \mathcal{B}$.

Proof. Let $\varepsilon_n := \frac{1}{n}$. Let $X = P_n \sqcup N_n$ be a Hahn decomposition for the signed measure $\mu - \varepsilon_n \cdot \nu$. If $P := \bigcup_n P_n$ is ν-nonnull, we are done because then $\exists n$ such that P_n is ν-nonnull, hence $\mu|_{P_n} - \varepsilon_n \cdot \nu|_{P_n}$ is a nonzero (unsigned) measure on P_n, so $\mu|_{P_n} \ge \varepsilon_n \nu_{P_n}$, as desired. Thus assume that $\nu(P) = 0$. It remains to observe that $N := X \setminus P = \bigcap_n N_n$ is μ-null (and hence $\mu \perp \nu$): $\mu(N) \le \mu(N_n) \le \varepsilon_n \nu(N_n) \le \varepsilon_n \cdot \underbrace{\nu(X)}^{<\infty} \xrightarrow[n\to\infty]{} 0$. $\square$

Now we deduce the Lebesgue's and the Radon-Nikodym theorems combined:

Lebesgue + Radon-Nikodym Theorem. Let μ and ν be σ-finite measures on a measurable space $(X, \mathcal{B})$. Then $\mu = \nu_f + \mu_0$, where $\mu_0 \perp \nu$ and $f : X \to [0,\infty]$ is a non-negative $\mathcal{B}$-measurable function. (Recall that $\nu_f(B) := \int_B f \, d\nu$.) This function f is unique up to ν-null sets. When $\mu \ll \nu$, i.e. $\mu_0 = 0$, then we call this f the Radon-Nikodym derivative of μ over ν, denoted $d\mu/d\nu$.

Proof. The uniqueness is HW, so we prove existence. As usual, by writing $X = \bigsqcup_n X_n$, where each $X_n \in \mathcal{B}$ and is both μ and ν finite, we may restrict to each X_n, so assume WLOG that both μ and ν are finite measures.

We aim to find a desired function f as follows: let

$$\mathcal{F} := \{ f : X \to [0,\infty] : f \text{ is } \mathcal{B}\text{-measurable and } \mu \geq \nu_f \}.$$

Claim. $\exists f \in \mathcal{F}$ such that $\int_X f \, d\nu = \sup \{ \int_X g \, d\nu : g \in \mathcal{F} \}$.

Pf. Note that $0 \in \mathcal{F}$ and $\mathcal{F}$ is closed under (finite) max: $f, g \in \mathcal{F}$ then $\max(f,g) \in \mathcal{F}$ because $\mu|_{\{f \geq g\}} \geq \nu_f|_{\{f \geq g\}}$ and $\mu|_{\{g \geq f\}} \geq \nu_g|_{\{g \geq f\}}$.

Let $f_n \in \mathcal{F}$ so that $\lim_n \int_X f_n \, d\nu = \sup \{ \int_X g \, d\nu : g \in \mathcal{F} \} \leq \mu(X) < \infty$. Replacing each f_n with $\max(f_0, f_1, \dots, f_n)$, we may assume that (f_n) is increasing.

Then $f := \lim_n f_n$ exists and by the MCT, $\int_B f \, d\nu = \lim_n \int_B f_n \, d\nu \leq \mu(B)$, so $f \in \mathcal{F}$ and $\int_X f \, d\nu = \lim_n \int_X f_n \, d\nu = \sup\{\int_X g \, d\nu : g \in \mathcal{F}\}$. $\square$ (Claim)

Now let $\mu_0 := \mu - \nu_f$ and we show that $\mu_0 \perp \nu$. Indeed, applying the previous lemma to μ_0 and ν, we see that if $\mu_0 \not\perp \nu$, then $\exists \, \varepsilon > 0$ and a ν-nonnull $A \in \mathcal{B}$ such that $\mu_0|_A \geq \varepsilon \cdot \nu|_A$. But then $f + \varepsilon \cdot \mathbb{1}_A$ contradicts the choice of f: $\mu - \nu_f \geq \nu_{\varepsilon \cdot \mathbb{1}_A}$, so $\mu \geq \nu_{f + \varepsilon \cdot \mathbb{1}_A}$, hence $f + \varepsilon \cdot \mathbb{1}_A \in \mathcal{F}$ but $\int_X f \, d\nu < \int_X (f + \varepsilon \mathbb{1}_A) \, d\nu$, a contradiction. $\square$

Cor. Let μ, ν be σ-finite measures on a measurable space $(X, \mathcal{B})$ and suppose $\mu \ll \nu$. Then for any $\mathcal{B}$-measurable $g : X \to \mathbb{R}$ that is ν-integrable or non-negative, we have

$$\int g \, d\mu = \int g \cdot \frac{d\mu}{\cancel{d\nu}} \cancel{d\nu}. \qquad (*)$$

Proof. By def, we have $(*)$ for all indicator functions $\mathbb{1}_B$, $B \in \mathcal{B}$, hence we have it for all simple functions by linearity, hence all $\mathcal{B}$-measurable non-negative functions by MCT, hence all integrable functions by decomposing into a difference of non-negative functions. $\square$

Cor (Chain rule). Let μ, ν, ξ be σ-finite measures on a measurable space $(X, \mathcal{B})$.

If $\mu \ll \nu$ and $\nu \ll \xi$, then $\mu \ll \xi$ and $\frac{d\mu}{d\xi} = \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\xi}$.

Proof. By the uniqueness of Radon-Nikodym derivatives, we just need to show that $\frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\xi}$ satisfies the defining property of $\frac{d\mu}{d\xi}$, i.e. that

$$\mu(B) = \int_B \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\xi} \, d\xi$$

for all $B \in \mathcal{B}$. But

$$\mu(B) = \int_B \frac{d\mu}{d\nu} \, d\nu \overset{\text{previous corollary}}{=} \int_B \frac{d\mu}{d\nu} \frac{d\nu}{d\xi} \, d\xi. \qquad \square$$

Cor. Let μ, ν be σ-finite measures on a measurable space $(X, \mathcal{B})$. If $\mu \sim \nu$, then

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu}\right)^{-1}.$$

Proof. Follows from the chain rule applied to $\mu \ll \nu \ll \mu$:

$$\underline{1} \equiv \frac{d\mu}{d\mu} = \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\mu}. \qquad \square$$

Remark. Why call the Radon-Nikodym derivative a derivative? It's a HW exercise to show that if a distribution F of a locally finite Borel measure μ on $\mathbb{R}$ is continuously differentiable, then $\frac{d\mu}{d\lambda} = F'$, where λ is Lebesgue measure.